INFLUENCE OF INITIAL VELOCITY ON TRAJECTORIES OF A CHARGED PARTICLE IN UNIFORM PARALLEL ELECTRIC AND MAGNETIC FIELDS

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Abstract: An exploratory study on the trajectory of a charged particle moving in parallel uniform electric and magnetic fields has been carried out where electromagnetic radiation of an accelerated particle is not considered. A general solution for the particle motion equation is derived analytically using a simple method by applying the second Newton's law to the Lorentz force acting on the charged particle. The trajectory is a circular helix with time-dependent pitch. Specific solutions are obtained by varying the initial particle velocity in the absence of electric field. The result shows two basic patterns of trajectories: circular and circular helix with constant pitch. Parameters such as radius and helical pitch for circular helix trajectory as well as radius and its center position for circle trajectory of cyclotron motion can be obtained.

Keywords: Trajectory, Helix, Pitch, Electric field, Magnetic field, Initial velocity

Introduction

The motion of a charged particle in uniform electric and magnetic fields has been discussed in many undergraduate physics textbooks, especially for a straight line trajectory of a charged particle with zero resultant of magnetic and electric forces in a velocity selector (Serway, 1996; Cutnell and Johnson, 2013; Halliday et al., 2011; Benson, 1996). The other frequent topic discussed in the textbooks is a circular trajectory of a charged particle moving in a plane at right angles to a uniform magnetic field (Serway, 1996; Cutnell and Johnson, 2013; Halliday et al., 2011; Benson, 1996; Griffiths, 1989) and if the particle starts with an additional velocity parallel to magnetic field then it moves in a helix (Serway, 1996; Halliday et al., 2011; Griffiths, 1989). Spiral paths of charged particles occur in a magnetic bottle as a plasma confinement (Serway, 1996; Halliday et al., 2011; Benson, 1996).

Photodetachment of H^- in electric and magnetic fields has been conducted theoretically by several authors, such as in crossed electric and magnetic fields (Peters and Delos, 1993), in crossed electric and magnetic fields near a metal surface (Wang, 2014), in crossed gradient electric and magnetic fields (Wang and Cheng, 2016), and in parallel electric and magnetic fields near metal surface (Tang et.al., 2016) in which classical motion equations of the detached electron are derived from the Hamiltonian governing the electron. Using simpler method, motion equations of a charged particle in uniform crossed electric and magnetic fields are obtained by applying the second Newton's law to the Lorentz force acting on the particle (Khotimah et al., 2017). This work explores the motions of a charged particle within uniform parallel electric and magnetic fields. Analytical formula is derived from a simple method (Khotimah et al., 2017) to obtain the motion equations and the trajectories which are influenced by initial velocity of the particle and the magnitude of electric field.

Theory

In this work, it is considered a charged particle q with mass m moves with velocity $\vec{v} = \left(v_x\hat{i} + v_y\hat{j} + v_z\hat{k}\right)$ in uniform electric field $\vec{E} = E_z\hat{k}$ and uniform magnetic field $\vec{B} = B_z\hat{k}$ from initial position $\vec{r}_0 = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$ and initial velocity $\vec{v}_0 = v_{x0}\,\hat{i} + v_{y0}\,\hat{j} + v_{z0}\hat{k}$ as shown in Figure 1. The fields affect the particle motion through the Lorentz force $\left(\vec{F}\right)$ (Serway, 1996; Cutnell and Johnson, 2013; Halliday et al., 2011; Benson, 1996; Griffiths, 1989).

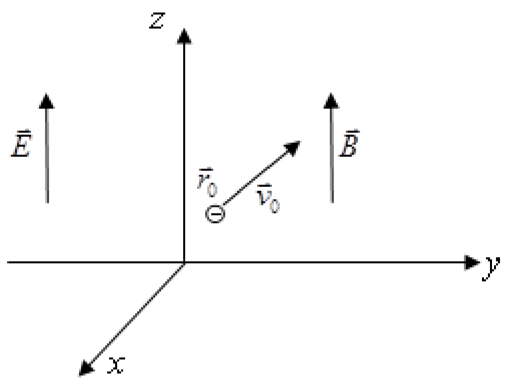

Figure 1. Schematic plot of a charged particle with initial conditions $\vec{r}_0 = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$ and $\vec{v}_0 = v_{x0}\,\hat{i} + v_{y0}\,\hat{j} + v_{z0}\hat{k}$ in uniform parallel magnetic and electric fields along the +z-axis

$$\vec{F} = qE_z\hat{k} + q\left(v_x\hat{i} + v_y\hat{j} + v_z\hat{k}\right)\times B_z\hat{k} \tag{1}$$

Applying the second Newton's law on the charged particle acted by the Lorentz force is

$$m\left(a_x\hat{i} + a_y\hat{j} + a_z\hat{k}\right) = \left(qv_yB_z\right)\hat{i} + \left(-qv_xB_z\right)\hat{j} + \left(qE_z\right)\hat{k} \tag{2}$$

Therefore, three differential equations are obtained.

$$\frac{d^2x}{dt^2} = \omega\frac{dy}{dt}, \qquad \frac{d^2y}{dt^2} = -\omega\frac{dx}{dt}, \qquad \text{and} \qquad \frac{d^2z}{dt^2} = \frac{qE_z}{m} \tag{3}$$

ω is the cyclotron frequency, at which the particle would orbit in the absence of electric field (Serway, 1996; Cutnell and Johnson, 2013; Halliday et al., 2011; Benson, 1996; Griffiths, 1989).

$$\omega = \frac{qB_z}{m} \tag{4}$$

The first two of equations (3) are coupled differential equations. By differentiating the first of equations (3) with respect to t and using the second equations (3) to eliminate $\frac{d^2y}{dt^2}$, we obtain a linear third order differential equation with constant coefficients,

$$\frac{d^3x}{dt^3} + \omega^2\frac{dx}{dt} = 0$$

The solution for $x(t)$ is obviously obtained.

$$x(t) = C_1\cos\omega t + C_2\sin\omega t + C_3 \tag{5}$$

Inserting solution (5) into the first of equations (3), we obtain $\frac{dy}{dt}$ and then it gives the solution for $y(t)$.

$$y(t) = C_2\cos\omega t - C_1\sin\omega t + C_4 \tag{6}$$

The third of equation (3) is for motion under constant acceleration so that the position of particle in z-axis as a function of time is

$$z(t) = C_5 + C_6\,t + \frac{1}{2}\frac{qE_z}{m}t^2 \tag{7}$$

Equations (5), (6), and (7) are a complete set of solutions for particle motion.

The initial condition of the charged particle is specified first. It starts at position $\vec{r}_0 = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$ and moving with initial velocity $\vec{v}_0 = v_{x0}\,\hat{i} + v_{y0}\hat{j} + v_{z0}\hat{k}$. These six conditions, i.e. $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$, $v_x(0) = v_{x0}$, $v_y(0) = v_{y0}$, and $v_z(0) = v_{z0}$, determine the constants C_1, C_2, C_3, C_4, C_5, and C_6. Thus, the equations of motion for the particle are

$$x(t) = -\frac{v_{y0}}{\omega}\cos\omega t + \frac{v_{x0}}{\omega}\sin\omega t + x_0 + \frac{v_{y0}}{\omega} \tag{8}$$

$$y(t) = \frac{v_{y0}}{\omega}\sin\omega t + \frac{v_{x0}}{\omega}\cos\omega t + y_0 - \frac{v_{x0}}{\omega} \tag{9}$$

$$z(t) = z_0 + v_{z0}\,t + \frac{1}{2}\frac{qE_z}{m}t^2 \tag{10}$$

Equations (8) and (9) are sinusoidal functions and they define the trajectory of the particle in *x*-*y* plane.

$$\left(x - \left(x_0 + \frac{v_{y0}}{\omega}\right)\right)^2 + \left(y(t) - \left(y_0 - \frac{v_{x0}}{\omega}\right)\right)^2 = \left(\frac{v_{x0}}{\omega}\right)^2 + \left(\frac{v_{y0}}{\omega}\right)^2 \tag{11}$$

Equation (11) is a formula for a circle in *x*-*y* plane of radius R

$$R = \sqrt{\left(\frac{v_{x0}}{\omega}\right)^2 + \left(\frac{v_{y0}}{\omega}\right)^2} \tag{12}$$

which is centered at point C $\left(\left(x_0 + \frac{v_{y0}}{\omega}\right), \left(y_0 - \frac{v_{x0}}{\omega}\right)\right)$.

Therefore, the general trajectory of the particle is a circular helix with radius R and pitch p.

$$p = v_z\left(\frac{2\pi m}{|q|B_z}\right) = \left(v_{z0} + \frac{qE_z}{m}t\right)\left(\frac{2\pi m}{|q|B_z}\right) \tag{13}$$

The pitch is along *z*-axis and is a function of time.

Results and Discussion

In this study, an electron ($m = 9.11\times10^{-31}\,\text{kg}$, $q = -1.60\times10^{-19}\,\text{C}$) is used as an example of a charged particle to show the particle motions. The electron moves in uniform magnetic field $\vec{B} = 1.0\times10^{-7}\,\hat{k}\,\text{T}$ and uniform electric field $\vec{E} = -2\times10^{-4}\,\hat{k}\,\text{N/C}$. The magnetic force provides the centripetal acceleration in the *x*-*y* plane and the electric field gives constant acceleration in the *z*-axis. For example, if the electron starts from initial position $\vec{r}_0 = (0\hat{i} + 0\hat{j} + 6\hat{k})$ m with initial velocity $\vec{v}_0 = (5.0\times10^3\,\hat{i} + 5.0\times10^3\,\hat{j} + 2.0\times10^4\,\hat{k})$ m/s then the equations (4), (8) to (10) give (in SI units):

$$\omega = \frac{qB_z}{m} = -1.76\times10^4 \tag{14}$$

$$x(t) = 0.285\cos(1.76\times10^4 t) + 0.285\sin(1.76\times10^4 t) - 0.285 \tag{15}$$

$$y(t) = 0.285\sin(1.76\times10^4 t) - 0.285\cos(1.76\times10^4 t) + 0.285 \tag{16}$$

$$z(t) = 6 + 2\times10^4 t + \frac{1}{2}3.51\times10^7 t^2 \tag{17}$$

The trajectory of the particle in *x*-*y* plane and the helical pitch are

$$(x + 0.285)^2 + (y - 0.285)^2 = (0.403)^2 \tag{18}$$

$$p(t) = 7.15 + 1.26\times10^4 t \tag{19}$$

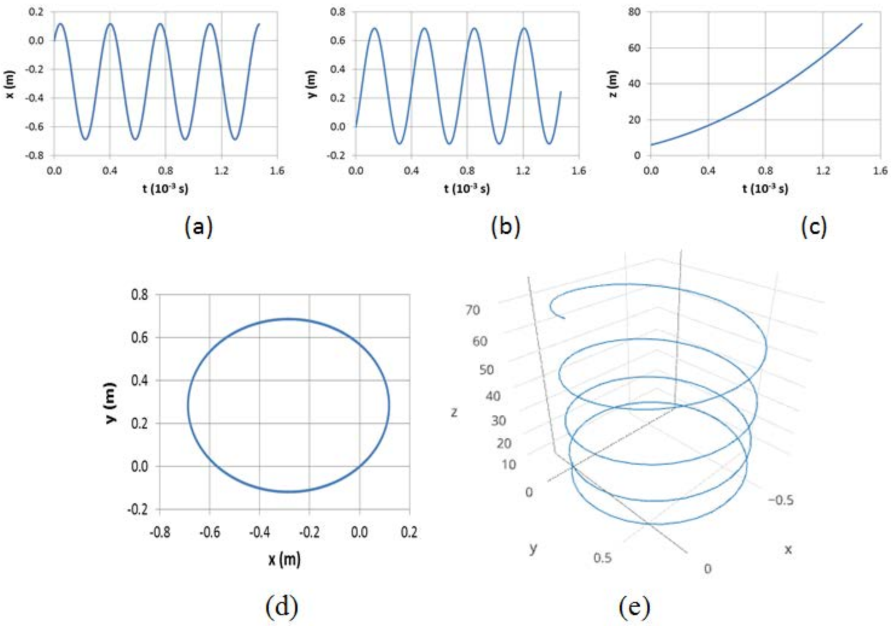

Figure 2. The motion of an electron from $\vec{r}_0 = (0\hat{i} + 0\hat{j} + 6\hat{k})$m with $\vec{v}_0 = (5.0\times10^3\,\hat{i} + 5.0\times10^3\,\hat{j} + 2.0\times10^4\,\hat{k})$m/s in $\vec{B} = 1.0\times10^{-7}\,\hat{k}$ T and $\vec{E} = -2\times10^{-4}\,\hat{k}$ N/C.
(a) *x*-component, (b) *y*-component, (c) *z*-component, (d) *x*-*y* plane, and (e) 3-D space

Figure 2 shows the particle motion in *x*-component, *y*-component, *z*-component, *x*-*y* plane, and circular helix trajectory with its pitch linearly changes with time. It can be seen clearly that the motion in *x*-component (Figure 2(a)) and *y*-component (Figure 2(b)) are sinusoid. According to equation (11), projection of the trajectory in *x*-*y* plane produces a circle with radius *R* of 0.403 m whose center at (–0.285m, 0.285m) as shown in Figure 2(d). The particle motion in *z*-component has a constant acceleration of 3.51×10^7 m/s^2 due to electric force and in this example it starts from z_0 = 6 m as presented in Figure 2(c). The trajectory of the particle is a tenuous circular helix as shown in Figure 2(e).

In this work, the influence of initial velocity $\vec{v}_0 = v_{x0}\,\hat{i} + v_{y0}\,\hat{j} + v_{z0}\hat{k}$ is studied. The *x* and *y*-component of initial velocity (v_{x0} and v_{y0}) influences the radius *R* of the circular helix according to equation (12). Meanwhile, the *z*-component of initial velocity (v_{z0}) influences the direction of helical motion. Figure 3 shows the influence v_{0z} of 2×10^4 m/s, 0, and -2×10^4 m/s in $\vec{v}_0 = (5.0\times10^3\,\hat{i} + 5.0\times10^3\,\hat{j} + v_{0z}\hat{k})$ on the trajectory. The trajectories are tenuous helix with their pitch changes with time due to constant acceleration in the *z*-component by the electric field as written in equation (10). The direction of helical movement in Figure 3(c) is initially toward *z*-negative and then finally to the *z*-positive as presented in Figure 3(d).

In the absence of electric field ($\vec{E} = 0$), the motion of the particle is influenced by the magnetic field alone so that the trajectory depends on the initial velocity of the particle as shown in Figure 4. If v_{0z} is constant then the trajectory is a circular helix with constant pitch (Figure 4a). This circular helix has radius *R* of 0.403 m and constant pitch of 7.15m. However, if $v_{0z} = 0$ then the trajectory is a circle in *x*-*y* plane (Figure 4b). The circular trajectory has the same radius *R* of 0.403 m. The center of circular trajectory is at point *C* (–0.285, 0.285, 6) m. This circular motion is usually only characterized by the radius of curvature in many physics textbooks (Serway, 1996; Cutnell and Johnson, 2013; Halliday et al., 2011; Benson, 1996; Griffiths, 1989).

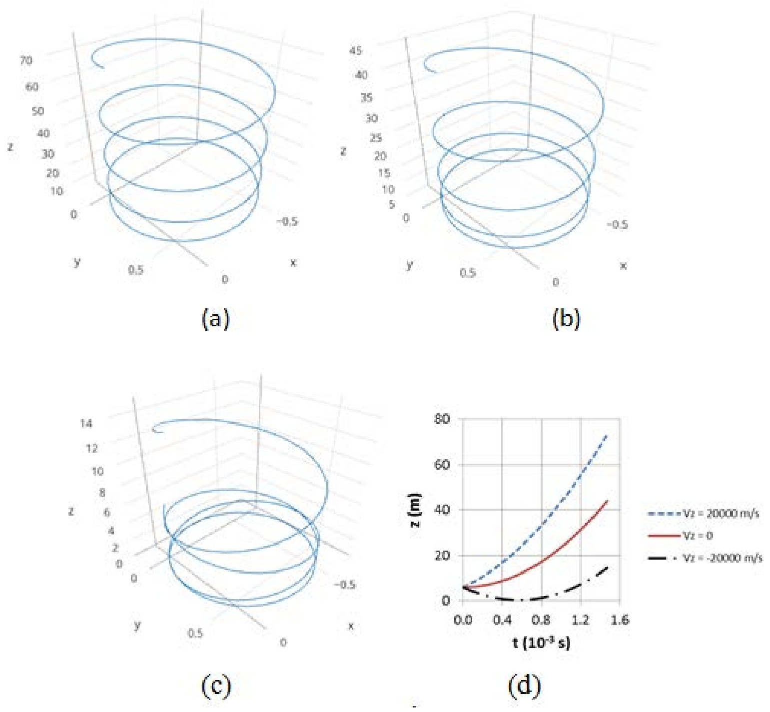

Figure 3. Circular helical trajectories of an electron in $\vec{B} = 1.0\times10^{-7}\,\hat{k}$ T and $\vec{E} = -2\times10^{-4}\,\hat{k}$ N/C.
It starts from $\vec{r}_0 = \left(0\hat{i} + 0\hat{j} + 6\hat{k}\right)$m with $\vec{v}_0 = \left(5.0\times10^3\,\hat{i} + 5.0\times10^3\,\hat{j} + v_{0z}\hat{k}\right)$m/s:
(a) $v_{0z} = 2\times10^4$, (b) $v_{0z} = 0$, (c) $v_{0z} = -2\times10^4$ and (d) the corresponding *z*-component

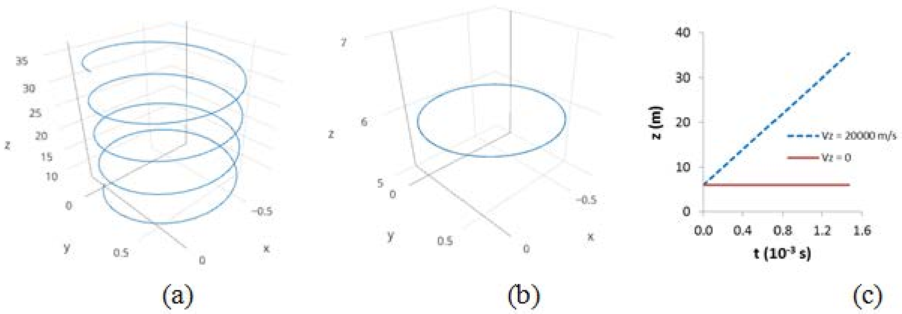

Figure 4. Helical and circular trajectories of an electron in $\vec{B} = 1.0\times10^{-7}\,\hat{k}$ T and $\vec{E} = 0$.
It starts from $\vec{r}_0 = \left(0\hat{i} + 0\hat{j} + 6\hat{k}\right)$m with $\vec{v}_0 = \left(5.0\times10^3\,\hat{i} + 5.0\times10^3\,\hat{j} + v_{0z}\hat{k}\right)$m/s:
(a) $v_{0z} = 2\times10^4$, (b) $v_{0z} = 0$ and (c) the corresponding *z*-component

The influences of initial velocity on circular trajectory of an electron in *x*-*y* plane starting from initial position $\vec{r}_0 = \left(0\hat{i} + 0\hat{j} + 0\hat{k}\right)$ within $\vec{B} = 1.0\times10^{-7}\,\hat{k}$ T and $\vec{E} = 0$ are shown in Figure 5. The influence of initial speed is shown in Figure 5a by taking four values, i.e. $\vec{v}_0 = n\times2.5\times10^3\,\hat{j}$ m/s; $n = 1,2,3,4$. and it shows longer radius of the circular trajectory for greater initial speed with $R = n\times0.142$ m; $n = 1,2,3,4$. The influence of initial velocity direction with the same speed is also presented in Figure 5b using initial speed $\left|\vec{v}_0\right| = 5.0\times10^3$ m/s and initial angle $\theta_i = i\dfrac{2\pi}{8}$; $i = 0,1,2,\ldots 7$. with respect to +*x*–axis which resulted different positions of the center $C\left(-0.285\sin\left(i\frac{2\pi}{8}\right), 0.285\cos\left(i\frac{2\pi}{8}\right), 0\right)$m $i = 0,1,2,..7.$ but with the same radii R = 0.285 m.

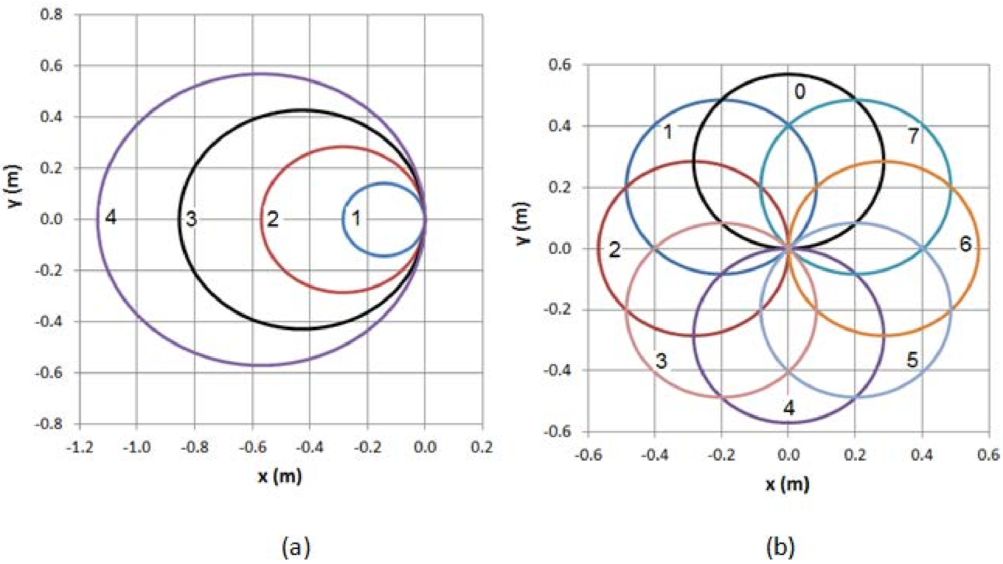

Figure 5. Circular trajectories of an electron in $\vec{B} = 1.0\times10^{-7}\,\hat{k}$ T and $\vec{E} = 0$.
It starts from origin $\vec{r}_0 = \left(0\hat{i} + 0\hat{j} + 0\hat{k}\right)$ with different initial velocities in *x-y* plane:
(a) influence of initial speed $\vec{v}_0 = n\times2.5\times10^3\,\hat{j}$ m/s; $n = 1,2,3,4.$
(b) influence of initial direction (angle) using speed 5.0×10^3 m/s: $\theta_i = i\frac{2\pi}{8}$; $i = 0,1,2,...7.$

Conclusions

This work has described the possibility trajectories of a charged particle in uniform electric field $\vec{E} = E_z\hat{k}$ parallel to magnetic field $\vec{B} = B_z\hat{k}$ influenced by initial velocity. The trajectory is circular helix with its pitch is a function of time if the particle has velocity component on the *x-y* plane ($\vec{v}_0 = v_{x0}\,\hat{i} + v_{y0}\,\hat{j} + v_{z0}\hat{k}$, $\vec{v}_0 = v_{x0}\,\hat{i} + v_{y0}\,\hat{j}$, $\vec{v}_0 = v_{x0}\,\hat{i}$ or $\vec{v}_0 = v_{y0}\,\hat{j}$), circular helix with constant pitch if the particle has velocity component in *z*-axis ($\vec{v}_0 = v_{x0}\,\hat{i} + v_{y0}\,\hat{j} + v_{z0}\hat{k}$, $\vec{v}_0 = v_{x0}\,\hat{i} + v_{z0}\hat{k}$, $\vec{v}_0 = v_{y0}\,\hat{j} + v_{z0}\hat{k}$) and $\vec{E} = 0$, and circle if the particle moves only in *x-y* plane ($\vec{v}_0 = v_{x0}\,\hat{i} + v_{y0}\,\hat{j}$, $\vec{v}_0 = v_{x0}\,\hat{i}$, $\vec{v}_0 = v_{y0}\,\hat{j}$) and $\vec{E} = 0$. The magnitude of initial velocity influences the radius and the direction of the initial velocity influences position of the center.

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