ROOFS, STAIRS & LINES: MIDDLE SCHOOL STUDENTS' STRATEGIES IN SOLVING STEEPNESS PROBLEMS

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ABSTRACT
Research shows that middle school students use a variety of strategies to solve proportion related problems (Hart, 1981). Mathematically, there is a connection between proportional reasoning and steepness, since steepness can be measured by the proportion which is the slope of a line. In this study, sixteen middle school students' solution strategies in solving problems regarding steepness are explored, based upon their abilities to solve proportion related problems. Two tests were administered to students: an adapted version of the Ratio and Proportion Test (Brown, et al., 1981) and a Steepness Test (Author, 2013). This article contributes to literature on early algebraic reasoning exploring applications of proportional reasoning.

INTRODUCTION

According to assessments administered by the International Association for the Evaluation of Educational Achievement, students internationally struggle with responding to real-life situations involving proportional reasoning and slope (Gonzalez, et al., 2008). Slope is an important concept because it represents a bridge between arithmetic and algebra (Lobato, 1996). The concept of slope draws upon knowledge of proportional reasoning and leads into the more general concept of rate of change, which is applied to increasingly complex types of functions in algebra and calculus. Steepness has been proposed as an intermediary concept to be introduced to bridge the conceptual gap between proportional reasoning and slope (Stump, 2001; Lobato & Thanheiser, 2002; Lobato & Ellis, 2010).

Through careful analysis of students' strategies as they work with proportions, we know a lot about students' development of knowledge about proportionality (Hart, 1984). And through careful analysis of students' strategies as they work with linear equations, we know a lot about the development of slope (Schoenfeld, Smith & Arcavi, 1993; Lobato & Thanheiser, 2002). However, there is little research that helps us understand how students make the conceptual leap from proportionality to slope. Mathematically, there is a clear connection between proportionality and slope, but the research base has not yet carefully examined this connection in terms of students' understandings.

The present paper explores this connection through an examination of students' strategies on a class of problems that are designed to bridge the concepts of proportionality and slope. As described in more depth below, we refer to these problems as "steepness" problems, in that the problems involve determining which of two situations (e.g., two roofs, two staircases, two lines) is steeper. As we argue below, understanding how students approach steepness problems helps the mathematics education community to understand whether and how students are using their knowledge of proportionality as they begin to learn about slope.

We begin by examining past studies that have focused on students' solutions to proportion-related, slope-related, and steepness-related problems; we then discuss the idea that using a ratio to measure steepness is a connecting concept between proportionality and slope.

STEEPNESS AS A CONCEPTUAL BRIDGE BETWEEN PROPORTIONALITY AND SLOPE

Many studies have found that the development of students' proportional reasoning follows a common learning trajectory (e.g., Kaput & West, 1994; Lamon, 1993). When first encountering contextual problems involving multiple quantities, some of which are related proportionally, there is evidence that many students have difficulty determining which of the quantities are relevant and which are irrelevant to the proportionality of the context (Karplus & Peterson, 1970). In addition, even when students are able to discern the relevant data, they may coordinate it in non-proportional ways, such as using only two of the three relevant pieces of information or using the three relevant pieces of information but in an additive way (Hart. 1981). Over time, students eventually learn to correctly coordinate the relevant quantities in problems whose solution processes require use of proportional reasoning (Lamon, 1993).

Research has also investigated the strategies used by students who are able to reason proportionally. For example, in order to solve missing value proportional reasoning problem (i.e., a problem that could mathematically be expressed using the equation $a/b=?/d$, where letters represent known quantities and the question mark represents an unknown quantity) and comparison proportional reasoning problems (i.e., a problem that could be mathematically be expressed as determining which of the two fractions, a/b or c/d is greater), Siegler (1976) found that, en route to thinking multiplicatively, students progress from using simple strategies that involve mostly additive reasoning to increasingly sophisticated strategies that take into account multiplicative relationships between quantities.

Studies have also determined that the numerical quantities involved in solving problems and the contexts in which the problems are situated may determine students' success on problems requiring proportional reasoning. For instance, problems asking for a comparison between four quantities which are related proportionally are easier for students to solve when there is some mathematical similarity between the quantities to be related, (Noelting, 1980), such as the numerators of 2/3 and 2/5 both being 2. Related, students tend to have greater success in solving comparison problems situated in contexts where the ratios are related by an integral factor (Lesh, Post & Behr, 1988), such as 2/2 and 6/6 which are related by a factor of the integer 3. In addition and more generally, students' familiarity with contexts can influence success on problem solving (Tourniaire & Pulos, 1985).

In addition to research studies' being conducted on students' development on proportionality, many studies have also investigated students' development of knowledge of slope and steepness. Researchers have observed that students tend to first develop a local perspective of slope prior to understanding slope as a global property of a line (Walter & Geruson, 2007; Schoenfeld, Smith & Arcavi, 1993). An understanding of slope at the local level involves identifying and interpreting slope for a specific segment of a line. However, students who understand slope only at the local level may think that the slope varies along a line segment as other points on the line are considered (Lobato & Siebert, 2002). For example, a student who is given the coordinates of three points on a line might separately compute the slopes of two separate line segments formed between the three points, instead of noticing that the slope of a line is constant throughout the line. A global understanding of a physically drawn line's slope entails knowledge that the slope of the line is constant, independently of the points selected from the line. An understanding of slope's meaning in a functional context involves understanding that slope is a constant rate of change between two quantities (Schoenfeld, Smith & Arcavi, 1993).

Much like the literature base on the development of strategies used to solve proportional reasoning problems, researchers have also observed that students' strategies used to solve slope problems often traverse a learning trajectory. This trajectory begins with the (erroneous) consideration of irrelevant information to determine slope and steepness. For instance, students may use the lengths of extraneous line segments in a diagram to determine steepness of a line (Lobato, 1996). Students may also consider the slope to be the y-intercept of the line (Moshkovich, 1996). Students' strategies progress to identifying the relevant information but using it in an incorrect way, such as finding the difference between the height and base length of a ramp rather than finding the ratio between the height and base length (Simon & Blume, 1994). Ultimately, when students understand the concept of slope, they are able to use relevant information to find the slope as a measure of steepness (Simon, 2006).

The existing literature base informs us that it is a difficult conceptual leap for students to use proportional reasoning in situations involving slope. When students are using additive reasoning to relate the relevant quantities instead of using proportional reasoning, it is likely that there is general confusion around at least one of the following ideas: proportionality, the proportionality that is inherent in slope, and the use of slope as a measure of steepness (Swafford & Langrall, 2000). Since an understanding of the concept of steepness is an intermediary step towards understanding that proportionality exists in situations regarding slope, this study examines the connections between students' abilities to solve proportional reasoning problems and students' strategies used to solve steepness problems, in an effort to understand when and why proportionality is used in slope work.

One way to help better connect proportionality with slope is through the use of finding a measure of steepness. In fact, after observing pre-service teachers working with slope-related problems, Simon and Blume (1994) postulated that using a ratio to determine the measure of steepness is a key developmental understanding, or fundamental prerequisite knowledge, of the concept of slope. Building off of this idea, after observing middle school students solving steepness-related problems, Lobato and Thanheiser (2002) outlined a series of steps which students would need to be able to do in order to accurately use a ratio as a measure of steepness: isolate the attribute that is being measured, determining which quantities affect the attribute and how, understanding the characteristics of a measure, and constructing a ratio. These empirically-based observations about the connections between proportionality, steepness, and slope are consistent with the mathematical connections between these concepts, which are all part of the multiplicative conceptual field that Vergnaud (1994) described as concepts whose relationships could be described multiplicatively.

In comparison to the abundant literature on students' strategies used to solve proportional reasoning problems and slope problems, there is very little literature on students' strategies used to solve steepness problems. An eighth grade teacher found that exposing her students first to the steepness of staircases helped these students later understand the formula for slopes of lines graphed on the coordinate plane, in other words, developing knowledge about steepness helped the students connect the geometric and algebraic conceptualizations of slope (Smith et al., 2013). In an activity which asked students to list how they would change a staircase to make it less steep, one eighth grade student suggested to make the stairs vertically shorter and to decrease their depth. While this student is indicating that both the vertical and horizontal dimensions of the staircase should be considered, the student demonstrates limited understanding of how these dimensions are coordinate with respect to steepness. If the original staircase has a vertical displacement of 1 foot and a horizontal displacement of 1 foot, following the student's suggestion to result in stairs that are steeper (e.g., 0.75 feet vertically and 0.5 feet horizontally), equally steep (e.g., 0.5 feet vertically and 0.5 feet horizontally), or less steep (e.g., 0.25 feet vertically and 0.5 feet horizontally). In order to better support this student to make a connection between steepness and slope, it would be useful for a teacher to know whether and to what extent the student is using proportional reasoning to think about steepness.

If we knew more about students' strategies used to solve steepness problems, in particular when and how students are using proportional reasoning to solve steepness problems, we might better understand students' difficulties with solving slope problems. The method by which this study investigates students' strategies is two-part. First, the study provides students with a variety of contexts in which steepness plays a role, to determine whether students are using relevant or irrelevant data to solve the problems. Secondly, the study provides a selection of structural difficulty levels inherent in each of the contexts to explore the degree to which proportional reasoning is a part of their solution processes.

This study compares students' strategies on steepness problems and proportion related problems in an effort to examine whether there is a positive relationship between students' solution methods on the two types of problems. Given that it appears that students who struggle with slope may have difficulties with understanding proportions (Lobato & Thanheiser, 2002), we hypothesize that students who have difficulties solving proportion related problems will also have difficulties solving steepness related problems.

During this study, we administered a pencil-and-paper Ratio and Proportion Test to students in grades 6 and 8. We then selected sixteen students to interview using a steepness test instrument to include a range of students' proportional reasoning levels as determined by the Ratio and Proportion Test. The strategies that the students used to solve the steepness problems are then analyzed in light of students' proportional reasoning levels, the contexts in which the steepness problems were situated, and the structural difficulty levels of the steepness problems.

RESEARCH QUESTIONS

The purpose of this study is to determine what relationships exist between students' strategies used to solve steepness problems and their abilities to solve proportion problems. More specifically, the study was designed to answer the following question:

1. What strategies do students, when classified by their proportional reasoning levels, use to solve problems involving steepness?
2. To what extent do students' strategies in solving steepness problems vary based on the:
   a. Context in which the steepness problem is situated?
   b. Structural difficulty of the problem?

METHODS

As described in more depth below, a survey consisting of two tests (a Ratio and Proportion Test and a Steepness Test) were administered to 16 students attending two middle schools.

Participants

The sample for the survey study consisted of 16 students in grades 6 and 8 who attended two private schools in the United States. Students were given the Ratio and Proportion Test in class prior to one-on-one interviews with the researcher after school. Interviewed participants were chosen such that students attaining all five levels of proportional reasoning were interviewed. The Steepness Test was given during the interview and participants were asked to think aloud as they solved the problems.

Procedure

Teachers administered participants the Ratio and Proportion Test in class. All students finished within the allotted time of 30 minutes. Participants did not receive incentives for participating in the study and were told that their participation would not impact their mathematics course grades. The author had a prior relationship with the school and the mathematics teachers; teachers mentioned to participants that this was part of a research study and they expected students to try their best.

The interviews were conducted by the author. Interviews were audiotaped and videotaped, and later transcribed. The interviews used the Steepness Test as the basis of a structured interview. The interviewer asked the participants to vocalize their thinking while they solved the problems. The interviewer only used verbalizations that were not likely to alter a participant's thinking process, as described by Ericsson and Simon (1984). Such verbalizations included encouraging the participant to keep speaking and asking the participant to articulate current and immediate past thoughts. The interviewer collected written work produced during the interview.

Instruments

To assess middle school students' levels of proportional reasoning, the Ratio and Proportion Test was developed from existing, similar assessments. The test items were adapted from the *Ratio and Proportion Test R* from the Concepts in Secondary Mathematics and Science (CSMS) Project (Brown, et al., 1981). The content validity and reliability of this test are described by Brown, et al. (1981). There were eight problem settings and a total of 20 problems on the Ratio and Proportion Test. Correct solutions received one point and incorrect solutions received zero points.

To assess middle school students' understanding of steepness, the Steepness Test was used (Author, 2013). The Steepness Test's construct validity and reliability analyses are described by Author (2013). The Steepness Test includes

24 problems that asked participants to determine which of two drawings was steeper. Each problem asked participants to compare the steepness of two objects and had three answer choices: 1) left object is steeper, 2) right object is steeper, or 3) the objects have the same steepness. There were three types of problem contexts: two situated the problem of steepness in a real-world situation and one presented it as a mathematical problem. Within each problem context there were eight problems, all grouped together. The two real-world situations were roofs and staircases. All drawings of roofs and staircases were shown on grid paper. The mathematical problems on the Steepness Test showed two lines in Quadrant 1, and each of them started at the origin. To solve these problems, participants needed to compare the steepness of the roofs, staircases, and lines. Lines were either explicitly shown, as in the case of the roofs or the lines on the coordinate plane, or not shown, as in the case of the staircases. All roofs, staircases, and lines were presented on coordinate grids with homogeneous axes. These contexts are either often used in slope problems in middle school mathematical textbooks or are often seen by students through their daily encounters with physical structures.
Within each set of eight problems (roofs, staircases, lines), the order of the problems based on structural difficulty level was randomly determined. The order in which the groups of problems based on context (roofs, staircases, lines) were arranged on the Steepness Test was not chosen at random, and may have had an effect on students' abilities to respond to the problems. Since the roof and the line problems present continuous data whereas the sets of staircase problems involve discrete data, the stairs were placed in the middle of the instrument. Hence, the eight roof problems were presented first, followed by eight staircase problems, followed by eight line problems.

Strategy Coding Scheme for Participant Responses to Steepness Test problems
Codes for strategies were created based upon the participants' responses on the Steepness Test. Strategies are the methods the participants used to solve the problems, and include implicit mathematical justifications as well as the explicit words spoken and actions taken. Strategies were assigned regardless of whether or not they were executed correctly. For example, a participant counted the correct number of horizontal and vertical boxes shown on the page, indicating that both dimensions were considered, but due to a counting error gave an incorrect answer. Some participants used more than one strategy to solve a problem, either to confirm the result of the first strategy or because they changed strategies. The response coded was the response used to produce the final answer. A total number of 384 problems were coded, produced from the sixteen participants' answers to the 24 Steepness Test problems.
The strategies that students used to describe steepness were coded using the following codes: 1) angles, 2) other, 3) irrelevant data, 4) area, 5) one measurement, 6) addition, 7) two measurements, 8) scaling, 9) norming, 10) ratio or rate. Each of the strategies is described below.
Angles The Angles strategy was coded when participants used the angles formed at the base of the staircases, at the corner of the roofs or at the base of the lines to determine steepness. Because all of the steepness problems were constructed on homogeneous axes of the same size, a comparison of angles could be used to obtain the correct answer. In some of the problems, the angles were very close in measure. Under these circumstances, it was impossible to accurately determine steepness using the Angles strategy, as measuring devices were not allowed. Participants whose responses were coded as "Comparing angles" by visual inspection; by using a benchmark angle such as 45 degrees, the horizontal line, or the vertical line; estimating angle degree measures; or by referring to parallel lines. For the staircase problems, some participants drew in auxiliary lines so that they could compare the steepness of those lines. Use of the angles strategy is beyond the scope of this article.
Other The first strategy reported in the results section of this article is the "other' strategy, coded as 0. This strategy was coded when no reasons were given or participants indicated they "guessed."
Irrelevant Data The Irrelevant Data strategy was coded when participants used data in the solution process that was not needed to determine steepness. For example, one participant compared the areas of the rectangles of the rectangular houses below each of the roofs, which is unrelated to the steepness of the roofs. Other irrelevant data given to determine steepness included the speed at which something would roll down a roof (since for two roofs that are equally steep, the longer roof will take longer for a ball to roll down), the difficulty of climbing a set of stairs (since the physical definition of work involves height alone) and the measures of the right angles forming individual steps in a staircase.
Area The Area strategy was coded when participants compared areas of roofs or the space between lines and the horizontal or vertical axes. This is a limited strategy because it will only yield a correct answer when at least one of the dimensions is held constant between two objects. None of the participants who used Area computed the area using a triangle area formula, rather, they visually compared the spaces or counted grid boxes that comprised the areas.
One Measurement The One Measurement strategy was coded when participants used only one dimension to determine steepness. Overwhelmingly, the measurement mentioned was either the horizontal or the vertical length, but it could also be the length of the diagonal formed by the side of a roof, the length of the line drawn from the top of the staircase to the base, or the length of the line drawn on the coordinate plane. This strategy is more advanced than the Irrelevant Data strategy because the measurement mentioned is partially useful in determining steepness.
Addition The addition strategy was coded when participants coordinated two measurements in an additive way. From literature on proportional reasoning, additive reasoning involves consideration of a difference between two

measurements. Participants who used additive reasoning often used it to incorrectly justify why two staircases had the same steepness.

Two Measurements The Two Measurements strategy was coded when participants mentioned or compared two measurements but did not explicitly relate them additively or using a ratio. This strategy is closer to using a ratio than the previously mentioned strategies because relating two measurements proportionally is one method of determining steepness. Both correct and incorrect responses were obtained using this strategy. Incorrect responses had mentions of comparisons of the vertical and horizontal measurements using terminology such as “taller” and “wider,” but arrived at an incorrect solution. Correct responses mentioned the two measurements or comparisons of the two measurements, and arrived at the correct solution. Participants who used two measurements only used the vertical and horizontal measurements, although using the length of the roof (eg, the hypotenuse of a right triangle formed by the vertical and horizontal measurements) could have been correctly used as well. Qualitative as well as numeric descriptions of the two dimensions were used.

Scaling The Scaling strategy was coded when participants said that one roof/staircase/line was a smaller or larger version of the other. Participants whose strategies were coded as Scaling often described an enlargement or shrinking. Because some of the slopes drawn were very close in value, it was sometimes difficult to determine whether one object was a smaller version of the other solely by visual inspection.

Norming The Norming strategy was coded when participants compared one measurement while holding another measurement constant. For example, a participant imagined that the two roofs to be compared were overlaid on top of each other, and concluded that at one particular horizontal distance, the left roof extends vertically higher than the right roof, the left roof must be steeper. When used correctly, Norming will always produce a correct response since its use is equivalent to the norming strategy used in proportional reasoning. Pictorially, norming by finding a common horizontal distance and comparing the vertical distances can be mathematically expressed as finding the common denominator of two slopes written as fractions and comparing the values of the numerators.

Ratio or rate The Ratio or Rate strategy was coded when participants indicated a proportion or rate. Some participants mentioned a numerical scale factor comparing two objects. When used correctly, the Ratio or Rate strategy will produce a correct response because slopes are being numerically compared.

The following table (Table 1) provides a summary of the strategies used, descriptions of these strategies, and some specific examples of their use.

Table 1. Descriptions and Examples of Coded Strategies.

Strategy	Description	Examples
Angles	Participant uses a visual comparison of angles, comparison to a benchmark angle or line, a comparison of two angles or parallel lines.	**Roofs:** A roof that is more level is less steep. **Staircases:** Stairs that look more straight are steeper. **Lines:** The line that is above the 45 degree line is steeper than the line underneath.
Other	Participant used a strategy not otherwise listed.	**Roofs:** One roof looks steeper. **Staircases:** When an auxiliary line is drawn, the stairs look the same steep. **Lines:** One line is “visually” steeper.
Irrelevant data	Participant takes into account information that does not need to be considered to solve the problem, or exclusively uses non-measured information that cannot be used exclusively to solve the problem correctly.	**Roofs:** The roof with a larger rectangular house base is steeper. **Staircases:** The stairs are equally steep because the angle of each stair corner is 90 degrees. **Lines:** One line would be easier to climb up.
Area	Participant compares area or space.	**Roofs:** The larger triangle formed by the two sides of the roof and the top of the rectangular house is steeper. **Lines:** A line with larger space underneath is steeper.
One Measurement	Participant compares only one measurement between objects, without mention or consideration of the other when it should be considered.	**Roofs:** The roof that is more vertical is steeper. **Staircases:** The higher stairs are steeper. **Lines:** The line with larger vertical length is steeper (without consideration of horizontal length).
Addition	Participant finds the difference between or within length measurements, and uses that to compare steepness.	**Staircases:** There is one square difference between two vertical lengths as well as one square difference between two horizontal lengths

		so the two objects are the same steep. **Lines:** Adding two onto the horizontal and vertical dimensions of a line creates a line that is equally steep.
Two measurements	Participant describes, finds or uses two lengths that need to be taken into account, but may not relate them additively or in a ratio.	**Roofs:** The roof that is wider and taller is steeper. **Staircases:** The stairs that have the same up-and-across are equally steep. **Lines:** The line that has larger area is steeper (where the triangles do not have one of the lengths held constant).
Scaling	Participant refers to an enlargement or shrinking of an object.	**Roofs:** One roof looks like a mini one of the other, so they are equally steep. **Staircases:** One staircase is a larger version of the other, so they are equally steep. **Lines:** One graph is a larger image than the other, so they are equally steep.
Norming	Participant holds one dimension constant while comparing the other, pictorially or numerically through length or area.	**Roofs:** If the horizontal length is held constant, a roof with a larger vertical length is steeper. **Staircases:** If two sets of staircases have the same vertical height, the one that takes up more horizontal space is steeper. **Lines:** The line with larger triangular area underneath is steeper if the horizontal base is the same.
Ratio or rate	Participant writes or says a representation indicating a ratio or proportion.	**Roofs:** One roof is double the other. **Staircases: "**[horizontal]4 units, 8 units, moving up 2 units, 1 unit… They're just half the size." **Lines:** One line goes up two units for each unit it goes across, and the other line goes up one unit for each unit across, so the taller one is steeper.

A total of 384 problems were coded, which came from 24 problems responded by each of the 16 interview participants. Reliability of the coding scheme was established by 97% agreement (or 75 of 77 codes) on 20% of the data (or 77 of 384 problems) coded between the researcher and another doctoral student in mathematics education. The first author coded the remaining 80% of the data.

One strategy code was assigned to each problem solution based upon the participant's final responses. Also, a record was kept of whether the initial and final responses of the participant were correct or incorrect. The correctness of the initial responses were used to compare interview participants' scores with those of the survey participants. The correctness of the final responses was used to compute success rates for the Two Measurements, Scaling, Norming, and Ratio or Rate strategies. The codes were aggregated by proportional reasoning levels. Charts were created that present the strategies used by participants of each proportional reasoning level, frequencies of correct and incorrect responses for each strategy for students in each proportional reasoning level, and success rates of some strategies by proportional reasoning levels.

RESULTS

Frequencies of strategies used by participants in each proportional reasoning level were determined and are reported in Figure 1. The use of the Angles strategy was omitted from this analysis since using a geometric way of comparing slopes is not a focus of this research question.

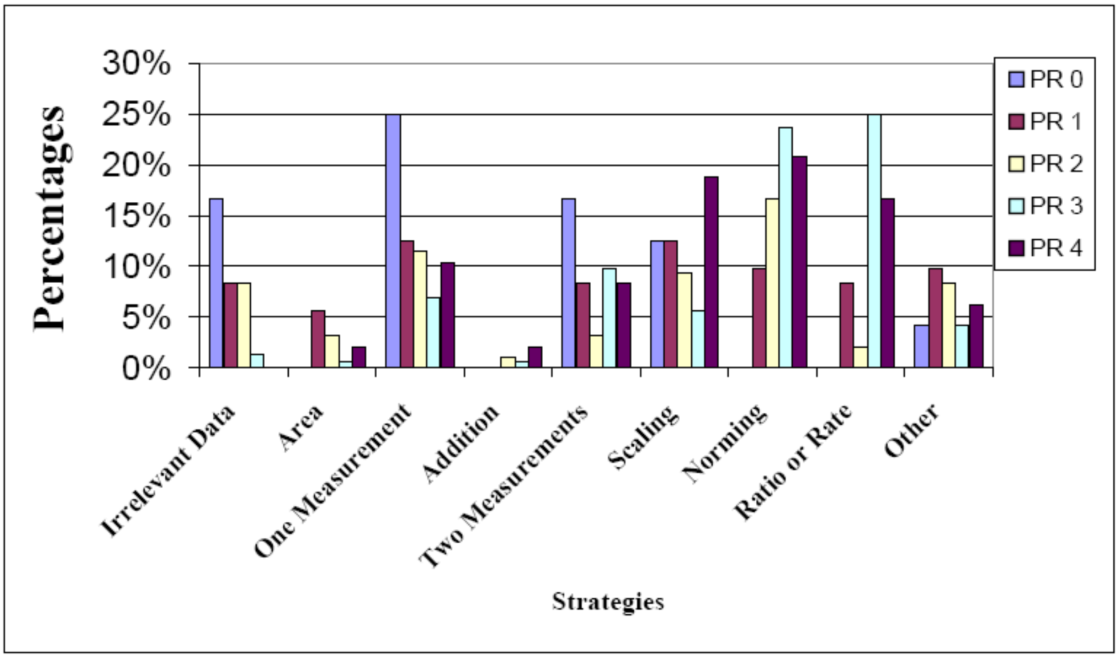

Figure 1. Participants' Solution Strategies by Proportional Reasoning Level.

From the interviews conducted, it appears that participants' strategies are somewhat related to their proportional reasoning levels. The participants who attained the higher proportional reasoning levels used the Norming and Ratio or Rate strategies more frequently, and the participants who attained the lower proportional reasoning levels used the Irrelevant Data, Area, and One Measurement strategies more frequently.

Four strategies used by interview participants involve some aspect of proportional reasoning: two measurements, scaling, norming, and ratio or rate. To investigate differences in performances using these strategies by participants in the five proportional reasoning levels, success rates were determined to indicate the percentage of correct responses yielded using each strategy. The success rates are presented in Table 2.

Table 2. Interview Participants' Success Rates for Four Strategies Most Related to Proportional Reasoning.

Strategies	Proportional Reasoning Levels				
	PR 0	**PR 1**	**PR 2**	**PR 3**	**PR 4**
Two Measurements	100%	50%	100%	90%	100%
Scaling	67%	67%	56%	60%	32%
Norming		70%	88%	88%	90%
Ratio or Rate		50%		76%	88%

The participants who attained the lower proportional reasoning levels had lower rates of success solving problems using Norming and Ratios or Rates, when these strategies were used. The Scaling strategy largely involved qualitative descriptions, and participants who attained the lower proportional reasoning levels had higher success rates solving these problems using Scaling. The Two Measurements strategy was used successfully for participants who attained Proportional Reasoning Levels 0, 2, 3, and 4, but participants who attained Proportional Reasoning Level 1 were only successful half of the time that they used this strategy.

To further investigate the relationship between participants' proportional reasoning levels and their solutions using the Two Measurements strategy, each of the problems coded as Two Measurements were also coded as qualitative or quantitative depending upon the type of descriptions that were given of the two measurements taken into account. While the Two Measurements strategy was used by research participants who attained proportional reasoning levels 0 through 4, the participants who attained higher proportional reasoning levels more often used quantitative descriptions of the two measurements, whereas participants who attained lower proportional reasoning levels more often used qualitative descriptions of the two measurements. The numbers of qualitative and quantitative descriptions for the Two Measurements strategy given by the participants in each of the proportional reasoning levels are shown below in Table 3.

Table 3.Interview Participants' Two Measurement Strategy Codes by Proportional Reasoning Level, Qualitative Description, and Quantitative Description.

Proportional Reasoning Levels	Qualitative Description	Quantitative Description	Total
PR 0	4	0	4
PR 1	6	0	6
PR 2	2	1	3
PR 3	6	8	14
PR 4	1	3	4
Total	19	12	31

According to Table 3, the participants who attained Proportional Reasoning Level 1 used only qualitative descriptions, and as Table 2 reported, half of these responses yielded incorrect responses. From Table 3, it was determined that the participants in who attained Proportional Reasoning Levels 3 and 4 used more quantitative descriptions than qualitative descriptions when they used the Two Measurements strategy, whereas participants who attained the Proportional Reasoning Levels 0, 1, and 2 used more qualitative descriptions than quantitative descriptions when they described the use of two measurements to compare steepness. Even though participants who attained Proportional Reasoning levels 0, 1, 3, and 4 had high success rates using the Two Measurements strategy (as shown in Table 2), Table 3 shows that participants who attained the higher proportional reasoning levels used more quantitative descriptions than participants who attained the lower proportional reasoning levels.

To examine the relationship between participants' proportional reasoning levels and the strategies they used on Steepness Test problems by context, percentages of Steepness Test strategies in each of the proportional reasoning levels were computed for each context. Table 4 shows the percentages of strategies used on the Steepness Test roof, staircase, and line problems.

Table 4. Interview Participants' Strategies by Proportional Reasoning Level and Steepness Test Context.

Staircase Problem Strategies	Proportional Reasoning Levels				
	PR 0	PR 1	PR 2	PR 3	PR 4
Irrelevant Data	37.5%	16.7%	12.5%	4.2%	
Area			6.3%		
One Measurement	50.0%	16.7%	12.5%	8.3%	25.0%
Addition			3.1%		6.3%
Two Measurements	12.5%	8.3%	3.1%	14.6%	12.5%
Scaling		8.3%	6.3%	4.2%	18.8%
Norming		4.2%	3.1%	14.6%	12.5%
Ratio or Rate		25.0%		35.4%	25.0%
Other		4.2%	21.9%	6.3%	

Roof Problem Strategies	Proportional Reasoning Levels				
	PR 0	PR 1	PR 2	PR 3	PR 4
Irrelevant Data		8.3%	9.4%		
Area				2.1%	6.3%
One Measurement	25.0%	8.3%	9.4%	4.2%	
Addition					
Two Measurements	37.5%	16.7%	6.3%	8.3%	6.3%
Scaling	25.0%	20.8%	18.8%	8.3%	37.5%
Norming		16.7%	21.9%	25.0%	18.8%
Ratio or Rate			3.1%	20.8%	12.5%
Other	12.5%	4.2%	3.1%		

Line Problem Strategies	Proportional Reasoning Levels				
	PR 0	PR 1	PR 2	PR 3	PR 4
Irrelevant Data	12.5%		3.1%		
Area		12.5%	12.5%	8.3%	6.3%
One Measurement				6.3%	6.3%
Addition				2.1%	
Two Measurements	12.5%	8.3%	3.1%	4.2%	
Scaling		8.3%	25.0%	31.3%	31.3%
Norming			3.1%	18.8%	12.5%
Ratio or Rate		16.7%	3.1%		
Other		20.8%		6.3%	18.8%

Table 4 shows the percentages of responses within each proportional reasoning level that participants used each type of strategy on the Steepness Test problems. On the staircase problems, 37.5% of the responses of the participant who attained Proportional Reasoning Level 0 used the irrelevant data strategy, whereas none of the participants who attained Proportional Reasoning Level 4 used the irrelevant data strategy on staircase problems. For roof problems, 37.5% of the responses by participants who attained Proportional Reasoning Level 4 used the scaling strategy, 18.8% used the norming strategy, and 12.5% used the ratios or rates strategy. In contrast, 25% of the responses by the participant who attained Proportional Reasoning Level 0 used the scaling strategy, whereas the norming and ratio strategies were not used at all to solve roof problems. Larger percentages of participants who attained Proportional Reasoning Levels 3 and 4 used the scaling and norming strategies on line problems than participants who attained Proportional Reasoning Levels 0, 1, and 2. Thus, more advanced strategies were used in each Steepness Test context by participants who attained higher levels of proportional reasoning.

To examine the relationship between participants' proportional reasoning levels and the strategies they used on Steepness Test problems by structural difficulty level, percentages of strategies used by participants who attained each of the five proportional reasoning levels were computed for each structural difficulty level. Table 5 shows the percentages of strategies used by participants who attained Proportional Reasoning Levels 0, 1, 2, 3, and 4 to solve Steepness Test problems in structural difficulty levels 1, 2, 3, 4, 5, 6, 7, and 8.

Table 5. Interview Participants' Strategies by Proportional Reasoning Level and Steepness Structural Difficulty Level

	Proportional Reasoning Levels				
Level 1 Problem Strategies	**PR 0**	**PR 1**	**PR 2**	**PR 3**	**PR 4**
Irrelevant Data		11.10%	8.30%		
Area					16.70%
One Measurement			33.30%	16.70%	50.00%
Addition					
Two Measurements	66.70%	22.20%	8.30%	16.70%	
Scaling					
Norming				5.60%	
Ratio or Rate				22.20%	
Other		22.20%		5.60%	16.70%

	Proportional Reasoning Levels				
Level 2 Problem Strategies	**PR 0**	**PR 1**	**PR 2**	**PR 3**	**PR 4**
Irrelevant Data	33.30%	11.10%	8.30%		
Area					
One Measurement			25.00%	11.10%	16.70%
Addition					
Two Measurements	33.30%		8.30%	11.10%	
Scaling					
Norming		11.10%		22.20%	66.70%
Ratio or Rate	0.00%	11.10%		16.70%	
Other		22.20%	16.70%	5.60%	16.70%

	Proportional Reasoning Levels				
Level 3 Problem Strategies	**PR 0**	**PR 1**	**PR 2**	**PR 3**	**PR 4**
Irrelevant Data	33.30%	11.10%	8.30%		
Area		33.30%		5.60%	16.70%
One Measurement	33.30%	11.10%		16.70%	16.70%
Addition					
Two Measurements					
Scaling			8.30%		16.70%
Norming		11.10%	33.30%	27.80%	16.70%
Ratio or Rate		11.10%		22.20%	
Other		11.10%	16.70%	5.60%	16.70%

	Proportional Reasoning Levels				
Level 4 Problem Strategies	**PR 0**	**PR 1**	**PR 2**	**PR 3**	**PR 4**
Irrelevant Data					
Area		11.10%			
One Measurement	33.30%	11.10%	8.30%	5.60%	
Addition					
Two Measurements		11.10%		16.70%	16.70%
Scaling	66.70%	33.30%	25.00%	22.20%	33.30%
Norming					16.70%
Ratio or Rate		11.10%		22.20%	16.70%
Other					

	Proportional Reasoning Levels				
Level 5 Problem Strategies	**PR 0**	**PR 1**	**PR 2**	**PR 3**	**PR 4**
Irrelevant Data	33.30%	11.10%	8.30%		
Area		11.10%			
One Measurement	66.70%		8.30%	5.60%	
Addition				5.60%	33.30%
Two Measurements				5.60%	
Scaling		33.30%	8.30%	5.60%	33.30%
Norming		11.10%	16.70%	27.80%	
Ratio or Rate		11.10%	16.70%	33.30%	33.30%
Other		11.10%	16.70%		

	Proportional Reasoning Levels				
Level 6 Problem Strategies	**PR 0**	**PR 1**	**PR 2**	**PR 3**	**PR 4**
Irrelevant Data		22.20%	16.70%	11.10%	
Area		11.10%			
One Measurement	66.70%	22.20%	16.70%	5.60%	
Addition			8.30%	5.60%	
Two Measurements	33.30%		25.00%	16.70%	16.70%
Scaling		11.10%	8.30%		16.70%
Norming		11.10%	16.70%	27.80%	
Ratio or Rate		11.10%		27.80%	50.00%
Other		11.10%	8.30%	5.60%	16.70%

	Proportional Reasoning Levels				
Level 7 Problem Strategies	**PR 0**	**PR 1**	**PR 2**	**PR 3**	**PR 4**
Irrelevant Data					
Area		11.10%	16.70%	5.60%	
One Measurement	33.30%	22.20%		5.60%	
Addition		11.10%			
Two Measurements			8.30%		16.70%
Scaling		22.20%	16.70%	11.10%	33.30%
Norming			33.30%	38.90%	16.70%
Ratio or Rate		11.10%		22.20%	16.70%
Other	33.30%				

	Proportional Reasoning Levels				
Level 8 Problem Strategies	**PR 0**	**PR 1**	**PR 2**	**PR 3**	**PR 4**
Irrelevant Data	33.30%		16.70%		
Area			8.30%		
One Measurement		11.10%		5.60%	
Addition					
Two Measurements	33.30%	22.20%	16.70%		
Scaling	33.30%		8.30%	5.60%	16.70%
Norming		33.30%	33.30%	38.90%	50.00%
Ratio or Rate				27.80%	16.70%
Other		22.20%	8.30%	16.70%	

* Use of the angles strategy is not reported in these tables

On problems with structural difficulty levels 3 through 7, participants who attained Proportional Reasoning Level 0 used the one measurement strategy to solve problems much more frequently than participants who attained higher levels of proportional reasoning. To solve the problems in structural difficulty levels 4 through 8, participants who attained Proportional Reasoning Level 4 predominantly used the two measurements, scaling, norming, and ratio or rate strategies. In general, more advanced strategies were used to solve the problems with higher structural difficulty by the participants who attained higher levels of proportional reasoning.

In structural difficulty levels 4 and 5, the objects to be compared were similar, thus the use of scaling was appropriate. According to Table 5, scaling was also used by participants of all proportional reasoning levels to solve problems with structural difficulties of 7 and 8, and since the objects to be compared were not similar, incorrect responses were obtained. The differences between the angles formed by the objects in these two levels were 3.18 degrees and 1.4 degrees respectively, and visually these small angular distinctions may be difficult to make in the absence of using other measurements.

DISCUSSION

One result of this study is the observation that solution strategies used to solve steepness problems are similar to those used to solve proportional reasoning problems. In this study, there was evidence of the incorrect use of addition, which took place when participants looked at additive differences between two measurements instead of looking at multiplicative relationships between the two measurements. This error, using additive thinking to describe proportional relationships, is well documented in the literature on proportional reasoning (Hart, 1981; Lamon, 2007; Simon & Blume, 1994). In the present study, one participant used the addition strategy to incorrectly explain that two staircases have the same steepness because of a common difference of 1 between the vertical and horizontal lengths. However, the use of the additive approach was relatively infrequent – less than 1% of the problems were solved using this strategy. Perhaps participants are less likely to use additive reasoning on comparison problems than they are on missing value problems, or perhaps the context of the problem influences participants' choice of strategy. Another explanation for the infrequent use of additive reasoning could be that the contexts that were given on the Steepness Test did not encourage additive thinking. Additional research is needed to investigate the relationship between the application of additive solution strategies to comparison problems involving steepness.

In the present study, another common non-proportional strategy employed by participants was the one measurement strategy. Participants who attained lower proportional reasoning used the one measurement strategy more frequently than did participants who attained higher proportional reasoning levels. This may imply that participants who are less able to reason proportionally may need further instruction before they are able to take into account the two dimensions that are required for understanding steepness as a ratio.

Another error exhibited by participants in this study was the use of irrelevant data. Consistent with Lobato's (1996) findings when interviewing participants about ramps, participants in this study found a myriad of irrelevant data to consider while comparing the steepness of roofs, lines and stairs, including finding the area of the rectangular house underneath a roof, observing that each stair was formed by a 90 degree angle, and using the number of steps in the staircase to describe steepness. This finding shows that in order to understand steepness, participants need to understand what characteristics of objects needed to be measured in order to determine steepness. The use of irrelevant data decreased as proportional reasoning levels increased. Participants who attained proportional reasoning level 4 did not exhibit the use of irrelevant data. Future research could see whether instruction in proportional reasoning affects participants' abilities to attend to relevant data when solving steepness problems.

Some of the proportional strategies that participants used to solve Steepness Test problems are similar to strategies often used to solve missing value proportional reasoning problems. The norming strategy was used on approximately 17% of the problems. Norming to solve comparison proportional problems has been described by Lamon (2007) as fixing one quantity in a proportional reasoning problem so that the other quantity could be compared independently of other quantities. On steepness problems, norming involves pictorially holding constant one dimension, either vertical or horizontal, and comparing the other dimension alone. For example, some participants superimposed one roof on top of another and compared vertical heights at the same horizontal distance to determine the steeper roof. The norming strategy is classified as a more advanced strategy than simply using one measurement. By holding one measurement constant, participants who used the norming strategy demonstrated understanding that it is necessary to consider both the vertical and horizontal dimensions. Participants who attained Proportional Reasoning Levels 3 and 4 used the norming strategy much more frequently than participants who attained Proportional Reasoning Levels 1 and 2. Future research could investigate whether being able to norm using proportions as well in physical situations leads to increased understanding of functional situations that could involve norming.

The scaling strategy involved a qualitative description that one object was an enlarged version of the other, or that one object was a smaller version of the other. The scaling strategy was relevant only for problems with structural difficulty levels 4 and 5, whose objects were similar to each other. As reported in Table 5, the scaling strategy was used by participants of all proportional reasoning levels on some of the Steepness Test problems in structural difficulty levels 6, 7, and 8. Success using the scaling strategy to solve steepness problems ranged from 32% to 67%, indicating that this strategy was not well understood or implemented. Future research could involve interviews during which participants who start using the scaling strategy are prompted to provide quantitative justifications for their qualitative observations, to gain a better understanding of what participants determine to be a measure of steepness.

In this study, participants who attained higher proportional reasoning levels identified ratios or rates such as scale factors more frequently than participants who attained lower levels of proportional reasoning. An example of a Proportional Reasoning Level 4 interview participant's use of a rate on Steepness Test line problems is a participant's labeling the axes distance and time, and then comparing speeds. The participant applied knowledge about distance-rate-time graphs that she learned from science class to steepness problems. The ratio or rate strategy was used by participants who attained Proportional Reasoning Level 1 at a success rate of 50%, indicating that these participants did not have a reliable way of implementing this strategy. Proportional Reasoning Levels 3 and 4 participants used the ratio or rate strategy more often than other participants, and were able to attain success rates of over 75% using this strategy. A future study could investigate students' understanding of the procedures involved in solving proportional reasoning problems and their abilities to apply this knowledge.

Another finding of this study is that there is a relationship between participants' proportional reasoning abilities and the strategies that they employ to solve steepness problems. In this study, participants with lower proportional reasoning abilities used non-proportional strategies more often, and participants with higher proportional reasoning abilities used more proportional strategies more frequently to solve steepness problems. Participants who attained higher levels of proportional reasoning also tended to have higher success rates on solving the problems using advanced strategies and tended to use more quantitative data to support their claims. It was also determined that participants who attained higher levels of proportional reasoning tended to use more advanced strategies to solve steepness problems by context and by structural difficulty level. Possible explanations for participants' differing performances on the Steepness Test contexts and structural difficulty levels will be offered in light of the results of other research studies that have been conducted. The strategies which the participants used to solve Steepness Test problems are discussed.

Several research studies show that the ability to reason proportionally is highly dependent upon context, and that some tasks facilitate students' reasoning proportionally more than others (Chletsos, De Lisi, & Turner, 1989; Tourniaire & Pulos, 1985). Researchers have also found that students' familiarity with contexts tends to help them solve proportional reasoning problems (Bright, Joyner, & Wallis, 2003). However, there is a dearth of literature that investigates students' abilities to reason about steepness in various contexts. Results of this study reveal that participants whose proportional reasoning abilities were higher used the two measurements, scaling, norming, and ratio/rate strategies to solve steepness problems more often than participants with lower proportional reasoning abilities. Thus, reasoning about steepness was dependent upon context and proportional reasoning abilities.

Students' familiarity with contexts may guide them to pay attention to relevant data from visual pictures. On roof and staircase problems, participants may have been unclear as to what physical features to look for in determining relative steepness. Based on the interview study, it was seen that some survey participants with lower proportional reasoning abilities compared the areas underneath the roofs or drew an incorrect auxiliary line for the staircases. Mitchelmore and White (2000) hypothesized that the sloping edges of a hill depicted in their diagrams helped their grades 2-8 research participants identify similarities between the hill and a standard angle. A similar effect may have taken place in the present study. The staircase problems did not explicitly contain lines whose steepness could be compared, whereas the roof and line problems did contain lines whose steepness could be directly compared. Additionally, roof problems contained more lines than necessary (e.g., the rectangular houses underneath the roofs) whereas the line problems only depicted relevant lines. The extra lines on the roof drawings confused some of the interview participants with lower proportional reasoning abilities and they unsuccessfully used the irrelevant data strategy in their solutions. Perhaps in the development of proportional reasoning abilities, students gradually become accustomed to sorting out relevant from irrelevant data.

In addition to an investigation of participants' performances on context, an investigation of participants' performance on structural difficulty levels was conducted. In studies that investigated the cognitive demands for solving tasks of various structural difficulties, researchers have found that taking into account four quantities simultaneously, which is required for proportional reasoning, is cognitively more complex than only taking into account one quantity (Halford, Andrews, Dalton, Boag, & Zielinski, 2002; Siegler, 1976). The findings of the present study are consistent with the findings of research on relational complexity. During interviews, the one measurement strategy was appropriately used by participants who attained Proportional Reasoning Levels 2, 3, and 4 to solve Steepness Test Level 2 problems, but this strategy was used heavily by the participants who attained Proportional Reasoning Levels 0 and 1 on other steepness problems for which taking into account only one measurement would not suffice.

Another finding that is consistent with relational complexity theory is that approximately 26% of the strategies used to solve Steepness Test problems by the interview participants were coded as the angle strategy. Since the angle is one quantity that can be considered alone to determine relative steepness, participants may have resorted to that comparison even when a visual comparison of angles may have been difficult, since the differences between the angle measures decreased as structural difficulties increased. This result suggests that participants may have had an easier time comparing one quantity rather than two. An extension of this study could explore other factors which might contribute to students' success on steepness problems, including students' intuitions about angles.

Prior to this study, no investigations were conducted to determine whether a relationship existed between participants' proportional reasoning abilities and their abilities to solve steepness problems. Streefland (1985) notes that the ability to reason proportionally can help students to solve problems in a variety of domains, such as physics (e.g., density, pressure), chemistry (e.g., concentrations), and biology (e.g., cross section of muscles and forces exerted). In a related study, Fischbein, Pampu, and Manzat (1970) also found that the ability to reason proportionally helps participants to solve problems in other domains. Fischbein and his research team taught 9 and 10 year old research participants how to compare ratios, and they were able to apply this knowledge to solve problems involving probabilities and chance.

This study suggests that in order for proportionality to be successfully used in slope work, it may be necessary for students to have a strong ability to reason proportionally, understand proportionality's relevance in solving steepness problems, and be able to correctly coordinate relevant data. This study also suggests that there may be a developmental progression in the use of strategies to solve steepness problems that can be influenced by proportional reasoning abilities. The interview study included at most six participants who attained each proportional reasoning level. Further study with a larger sample size is necessary, however, before further generalizations can be made about participants' solution strategies on steepness problems based upon their proportional reasoning levels.

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